Reg. No. [ ]

MTH 504

Third Semester M.Sc. Degree Examination, December 2018/January 2019
MATHEMATICS
Multivariate Calculus and Geometry
(Choice Based Credit System – New Syllabus)

Time : 3 Hours Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. No additional sheets will be provided for answering.*
*3) **Use** of scientific calculator is **permitted**.*

1. a) Identify geometrically and sketch the level set $F^{-1}(C)$, where $F(x, y, z) = (z^2 - x^2 - y^2, 2x - y)$ and $C = (1, 2)$. From your sketch determine whether the level set is open, closed, bounded, compact.

   b) Find the tangent plane and normal line to the set of points satisfying $y^2 + z^2 - 2x^2 = 2$ and $xyz = 2$ at the point $(1, \sqrt{2}, \sqrt{2})$.

   c) Let $S_1 = \{(x, y, z) \in \mathbb{R}^3 : y = f(x)\}$ denote a cylinder and S_2 denote the level set $z^2 + 2zx + y = 0$. If S_1 is tangent to S_2 at all points of contact find f. **(5+5+4)**

2. a) State and prove the theorem on method of Lagrange multipliers to find the maxima/minima of a real valued function with constraints.

   b) Classify the non-degenerating critical points of
   $f(x, y) = (2 - x)(4 - y)(x + y - 3)$. **(8+6)**

3. a) Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ denote a continuous vector filed on the connected open set U of $\mathbb{R}^n$. Then prove that, F has a scalar potential if and only if for any two points A and B in $\mathbb{R}^n$ and any two piecewise smooth directed curves Γ_1 and Γ_2 joining A and B, we have $\int_{\Gamma_1} F = \int_{\Gamma_2} F$.

   b) Find the length of the curve parametrized by $P(t) = (2\cosh 3t, -2\sinh 3t, 6t)$, $0 \leq t \leq 5$.

   c) Obtain the unit speed parametrization of the curve defined by;
   $t \to (e^t \cos t, e^t \sin t, e^t)$, $t \in [0, 1]$. **(7+3+4)**

4. a) Explain the geometric interpretation of the curvature of a curve in $\mathbb{R}^2$.

   b) If $P : [a, b] \to \Gamma$ is a unit speed parametrized curve, prove that $\langle P' \times P'', P''' \rangle = \kappa^2 \tau$ Further if $\tau \neq 0$, then show that $\tau = \dfrac{\langle P' \times P'' \times P''' \rangle}{\langle P'', P'' \rangle}$.

   c) Let Γ is a directed curve in $\mathbb{R}^3$ with positive curvature at all points. Then prove that Γ is a plane curve if and only if its bi normal B(t) is constant for all t. **(4+4+6)**

5. a) State and prove Green's theorem.

   b) Obtain a parametrization of Torus of major radius b and inner (minor) radius a.

   c) Find the surface area of the paraboloid $z = x^2 + y^2$ which lies between the planes $z = 0$ and $z = 4$. **(6+4+4)**

6. a) Find $\iint_S F$ where $F(x, y, z) = (1, 2, 3)$ and S is a triangle with vertices (1, 0, 0), (0, 2, 0) and (0, 0, 3) oriented so that the origin is on the negative side.

   b) Verify the Stokes' theorem for the portion S of the surface $z = \tan^{-1}(y/x)$, which lies inside the cone $x^2 + y^2 = z^2$ and between the planes $z = 0$ and $z = 2\pi$, by using the vector field $F(x, y, z) = (xz, yz, -x^2 - y^2)$. **(6+8)**

7. a) If $V = \{(x, y, z) : 0 \leq x \leq 1, 1 \leq y \leq 5, 2 \leq z \leq 3\}$, then evaluate;

   $$\iiint_V x^2yz^2 \, dx \, dy \, dz.$$

   b) Find the volume of the region of $\mathbb{R}^3$ bounded by the plane $z = 3 - 2y$ and the paraboloid $z = x^2 + y^2$.

   c) State Gauss' divergence theorem. Let V be the solid cylinder $\{(x, y, z) : x^2 + y^2 < 1, 0 < z < 1\}$ and $F(x, y, z) = (1 - (x^2 + y^2)^3, 1 - (x^2 + y^2)^3 x^2z^2)$. Use divergence theorem to evaluate $\iiint_V \text{div}(F) dx \, dy \, dz$. **(2+6+6)**

8. a) At an umbilic point p of a surface S, prove that;

   i) $K(p) > 0$ if and only if S is shaped like a sphere near p.

   ii) $K(p) = 0$ if and only if S is very flat near p.

   b) Define the geodesic curvature of a directed curve Γ on an oriented surface S. Show that, a parameterized curve $P : [a, b] \to \Gamma \in S$ is geodesic if and only if it has a constant speed and zero geodesic curvature. **(7+7)**